Ion Containment with Magnetic Bottles

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Abstract

Ion containment is important to several branches of science, including physics and chemistry. Using vPython, this paper explores the motion of charged particles in a magnetic bottle, a type of magnetic mirror created by two current-carrying coils. The magnetic field is calculated and displayed inside of the bottle, and trajectories of ions that do and do not get trapped are shown. The velocity criteria of trapped particles is explored as well as other initial conditions (such as charge and initial position) that determine whether or not an ion gets trapped.

Introduction and Motivation

A magnetic mirror is magnetic field configuration in which charged particles are reflected from areas of strong magnetic field to areas of weak magnetic field. One common type of a magnetic mirror is a magnetic bottle. In a magnetic bottle, two current-carrying coils, separated by a small distance, are oriented in parallel planes. The current in each coil runs in the same direction, and the resulting magnetic field is capable of containing charged particles.

The act of trapping charged particles is very important to several branches of science, including physics and chemistry. Magnetic bottles and other ion traps allow for the brief containment and study of plasma, which is hot enough to melt any physical container. This has led to many practical uses, such as neon lights and microelectronic etching, and also more experimental research like fusion energy. Moreover, the Van Allen radiation belts, which trap harmful radiation from the sun, are essentially magnetic mirrors created by the earth's magnetic field. The motion of ions in magnetic bottles gives insight into how these radiation belts work.

This paper, more qualitatively than quantitatively, explores the field created by a magnetic bottle and the motion of a charged particle inside the bottle. Specifically, it examines the trajectory of a trapped ion; explores how this trajectory is affected by initial conditions of the ion, such as velocity and charge; and addresses how long an ion can be contained. These traits are also compared and contrasted to ions that are not trapped.

Theoretical Background

Motion of Charged Particles in Magnetic Fields

Experimental evidence shows that the force on a charged particle moving in a magnetic field exhibits the following: the force is proportional to the particle's velocity, the force is proportional to the particle's charge, and the direction of the force is perpendicular to both the velocity of the particle and the direction of the magnetic field. Symbolically, the equation

$$\vec{F} = q\vec{v} \times \vec{B} \tag{1}$$

gives the force on a particle with charge q and velocity $\vec{v}$ in a magnetic field $\vec{B}$. The direction of this force can be found by the right-hand rule for a positively charged particle; the direction is opposite that given by the right-hand rule for a negatively charged particle.

In a uniform magnetic field (i.e. a magnetic field with only one constant component), an moving ion will undergo uniform circular motion, given that its velocity is not parallel to the field. This is because the force on the ion will be everywhere perpendicular to its direction of travel (radially inward)—its direction will constantly change, but its velocity will not. It can be shown that the radius, r, of an ion's circular trajectory is

$$r = \frac{mv_\perp}{qB}, \tag{2}$$

where m is the mass of the ion and $v_\perp$ is the component of its velocity perpendicular to the field.

In general, as in the case with magnetic bottles, the magnetic field will be a function of position, and so the force on the particle will depend on the value of the field at that point. The next section shows how to calculate such a field.

Given the force on a particle, a closed-form expression for the velocity and final position of the particle could be given. However, it is sometimes more useful, especially in programming, to iteratively calculate the position. This can be done by updating the force on the ion, given by equation 1 for a charged particle in a magnetic field. Then, by Newton's second law $\vec{F} = \frac{d\vec{p}}{dt}$, so in a small time segment Δt, $\Delta\vec{p} = \vec{F}\Delta t$, and

$$\vec{p} = \vec{F}\Delta t + \vec{p}_0. \tag{3}$$

The velocity of the particle is then

$$\vec{v} = \frac{\vec{p}}{m}, \tag{4}$$

where m its mass. And, finally, the new position is given by

$$\vec{r} = \vec{v}\Delta t + \vec{r}_0. \tag{5}$$

Magnetic Fields

Just as only moving ions experience forces in a magnetic field, only moving ions generate magnetic fields. For a single ion in vacuum with charge q and velocity $\vec{v}$, the magnetic field produced at a distance r away from the ion is given by

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{q\vec{v} \times \hat{r}}{r^2}, \tag{6}$$

where $\mu_0 = 4\pi \cdot 10^{-7}$ T $\cdot$ m/A is a constant called the permeability of free space. The direction of $\vec{B}$ at the field point can also be evaluated with the right-hand rule.

An important principle of magnetic fields is that the total magnetic field at a point is equal to the sum of each source of magnetic field. This is called the superposition principle of magnetic fields.

The magnetic field of a current carrying wire can be derived from the above equation by considering a small segment of wire $\vec{dl}$ (the direction is in the same direction of the current) with charge dQ. If there n moving particles with charge q per unit volume, then

$$dQ = nqAdl,$$

where A is the cross-sectional area of the wire. All moving charges in the small segment are thus equal to one moving charge, dQ, traveling at the drift velocity $\vec{v}_d$ of the charges. Then,

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{dQ\vec{v}_d \times \hat{r}}{r^2} = \frac{\mu_0}{4\pi}\frac{nqA\vec{v}_d dl \times \hat{r}}{r^2} = \frac{\mu_0}{4\pi}\frac{Id\vec{l} \times \hat{r}}{r^2},$$

since $nqA\vec{v}_d$ is the current, I. In order to get the total magnetic field at position $\vec{r}$, we integrate over the entire current carrying wire L:

$$\vec{B} = \int_L \frac{\mu_0}{4\pi}\frac{Id\vec{l} \times \hat{r}}{r^2}. \tag{7}$$

The above equation is known as the law of Biot and Savart.

Calculations

Setup

In vPython, the magnetic bottle is modeled as two parallel coils (circles) of radius 0.01 m in the x-y plane, separated by a distance 0.025 m. H^+ is considered for the charged particle, which is represented as a small sphere. The path that the ion takes can be seen by the trail it leaves. Due to the very small mass of H^+, the force of gravity is neglected throughout.

Calculating the Magnetic Field

Seeing the magnetic field lines is very important for understanding the path an ion takes. This section is dedicated to doing just that. For the magnetic bottle, the current in each coil is running in the same direction. Following the law of Biot and Savart, we first pick a point inside of the bottle. At that point, we divide the current-carrying coil into small segments $d\vec{l}$ and calculate the small magnetic field $d\vec{B}$ according to equation 6.

We continue around the entire length of the coil, summing up each small segment in order to get a numerical approximation of the integral in equation 7. The same process is done on the second coil, and the total field at that point is found using the superposition principle of magnetic fields. After doing this process at multiple points, we arrive at the magnetic field in Figure 1.

All vectors have a large downward component, which is expected for the given direction of the current. Near the coils, the field is strongest, and near the center of the bottle, the field is weakest. On the outer edge of the bottle, along the circumference of the coils, the field lines begin to curve outward. However, the field is fairly uniform near the center, with vectors directed almost straight down.

Ion Trajectories

For each of the following trajectories, the position of the ion is calculated using the iterative process described in the Theoretical Background Section. Initially, due to the uniformity of the field in the center, I thought that a particle would best be trapped in the plane parallel to the coils crossing through the center of the bottle. Indeed, if the particle started inside the bottle in this center-plane, then it would simply undergo circular motion and would thus be trapped. However, the path it takes as it approaches the bottle is more complex. Figure 2 shows the trajectory that the particle undergoes.

Magnetic Field of Magnetic Bottle

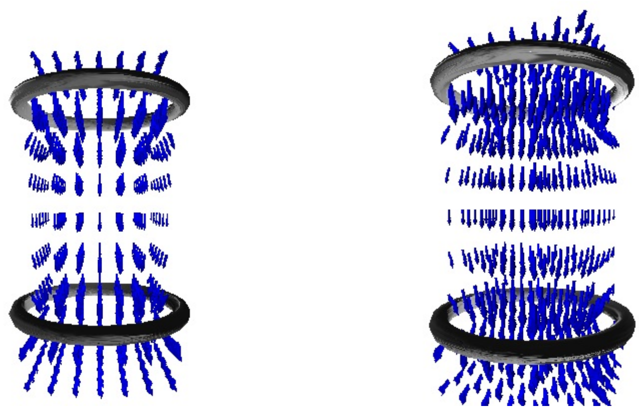

Figure 1: Two angles of the field. Current is clockwise, as seen from above. (Vectors are scaled.)

Spiral Trajectory

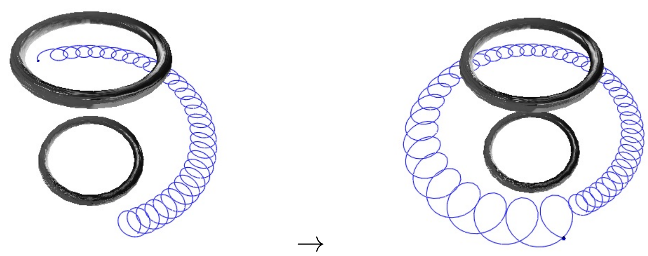

Figure 2: The path of an ion in the plane of the coils halfway between each.

Here, the particle has initial speed of about 10^6 m/s in a plane parallel to the coils; it is traveling into the page and to the right (see picture on left). The magnetic bottle causes it to spiral counterclockwise around the "outside" of the bottle, with successive spirals increasing in radius. Because of this, the particle eventually spirals away from the coils and escapes the magnetic bottle. It should also be noted that an extremely large magnetic field, about 25 T where the ion was traveling, was required for this trajectory.

Although this trajectory is rather interesting, it is not the best way that a magnetic bottle can trap an ion. It turns out that the maximum magnetic field strength is not the important criteria of a magnetic bottle. Rather, it is the ratio of the strongest point B_{max} to the weakest point B_{min} in the particle's path (see equation 8). In the bottle, the strongest point is somewhere near center of the coils, and the weakest point is somewhere in the middle of the bottle, as can be seen in Figure 1. For this reason, it is most desirable to have the ion pass through these points, and thus be directed "into the bottle," through either coil. The trajectory of such a particle is shown in Figure 3.

The ion, once it reaches the bottle, begins to spiral downward, with the radius of spirals getting larger toward the center of the bottle, then decreasing near the other coil. This can be understood by the field and equation 2—the field is stronger near either coil and weaker near the center. Quite surprisingly, when the ion reaches the other coil, it spirals rapidly and then recoils, traveling in the

Trapped Ion in a Magnetic Bottle

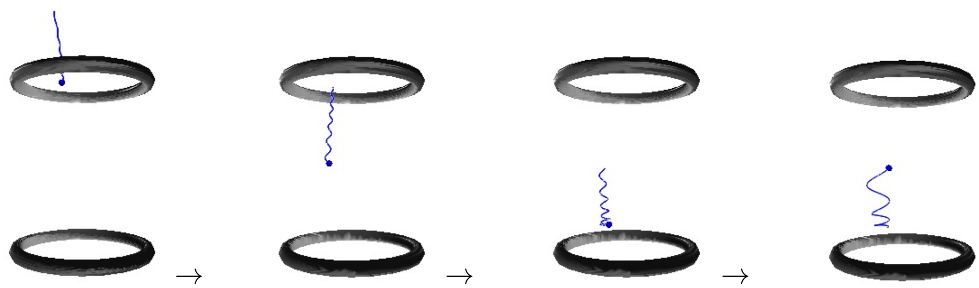

Figure 3: The path of an ion with appropriate velocity directed into the bottle.

opposite direction. This is why configurations like this are termed magnetic mirrors. Ideally, the ion will repeat this process indefinitely, bouncing back from coil to coil, and thus will be trapped in the magnetic bottle.

In this same setup, the ion, with the same speed, was given different initial conditions and the trajectory was evaluated. The following was found:

1. The sign of the charge did not affect whether the ion was trapped; it only changed the direction of spiraling. However, the magnitude of the charge did affect it.

2. Whether the particle was directed up or down into the bottle did not matter.

3. The sign of the ion's perpendicular component of velocity did not matter.

While the above are true for the particular velocity that was trapped, there is a velocity criterion for trapping. Figure 4 exhibits this.

Ion Velocities that are not Trapped

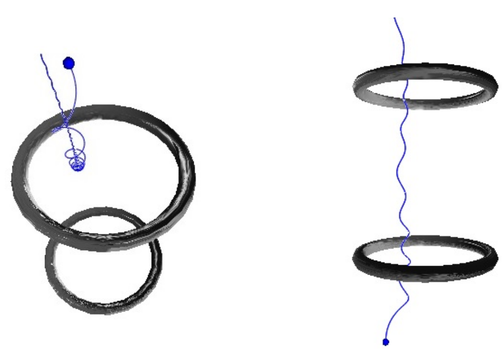

Figure 4: **a)** An ion that is reflected before entering the bottle. **b)** An ion that never gets reflected from the second coil.

With the perpendicular component of velocity (to the initial magnetic field), $v_\perp$, kept the same for the trapped ion, the parallel component, $v_\parallel$ was varied. It was found that some speeds (of $v_\parallel$) were too slow—the ion bounced off the first coil and never made it into the bottle. Other speeds were too fast—the ion never bounced off the second coil and escaped the bottle.

In general, it turns out that once the ion is inside the magnetic bottle, the velocity criteria of a reflected particle is

$$\frac{|v_\perp|}{|v_\parallel|} \geq \frac{1}{\sqrt{R_m}}, \tag{8}$$

where $R_m = \frac{B_{max}}{B_{min}}$ is called the mirror ratio, and where $v_\parallel$ and $v_\perp$ are measured at the weakest point of the magnetic field, B_{min}. [1]

Rewriting this equation, we see that

$$\frac{|v_\perp|^2}{|v_\parallel|^2} \geq \frac{B_{min}}{B_{max}}. \tag{9}$$

This means that, for a given magnetic bottle, there is a large range of ion speeds that can be trapped, provided that the ratio of the ions' perpendicular and parallel component of velocity in the weak field region meets the above requirement. While experimenting in vPython, large speeds, about 10^6 m/s, were chosen so that the program would run faster.

If an ion's velocity does not meet this criteria, then it will escape the bottle. An example of this is part (b) of Figure 5. (In part (a), the ion never "enters" the bottle, and so it does not have a chance to escape. This can be regarded as a separate velocity condition, whereas equations 8 and 9 apply to an ion already in the bottle.) Such an ion is said to be in the loss cone of the magnetic bottle, shown in Figure 5. [2]

The Loss Cone

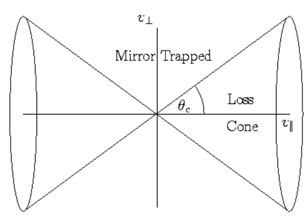

Figure 5: Graphical representation of the velocity dependence of trapped ions in a magnetic bottle.

In practice, all ions will eventually escape a magnetic bottle, regardless of their velocity. This is because a trapped ion can gain kinetic energy when it is moving back and forth along its path in the magnetic bottle. This increases the speed of the particle, which affects both the parallel and perpendicular components of its velocity. Thus, while the ion may be trapped with its initial velocity, the same is not true for the ion's velocity at a later time, and the ion can escape.

In Figure 6, the particle reflects off the coils about three times before its trajectory becomes chaotic. After this, it stays in the bottle briefly in a trajectory that takes it around the circumference of the coil, then it escapes completely.

The graph, shown in Figure 7, of the kinetic energy of the ion reveals that its speed is indeed increasing. The speed increases at all times inside the bottle, but it does so most rapidly when the ion is near either coil. This is because the magnetic field is stronger near either coil, and so the force is stronger, according to equation 1. After it escapes the bottle, its speed and kinetic energy remain approximately constant because the magnetic field, and thus the force on the ion, is very small.

[1] University of Glasgow Astronomy and Astrophysics Group, PowerPoint by Lyndsay Fletcher; http://www.astro.gla.ac.uk/users/lyndsay/TEACHING/NLPII/Lecture4_06.pdf

[2] Image from MIT, "Introduction to Plasma Physics" notes by Ian Hutchinson; http://silas.psfc.mit.edu/introplasma/

Eventual Escape of Ions

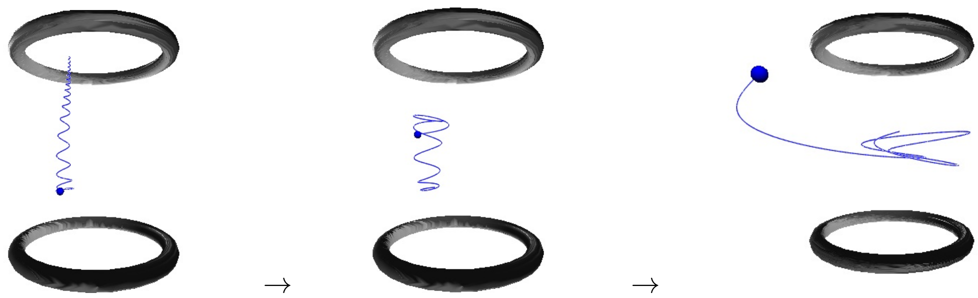

Figure 6: An ion that is trapped will eventually escape. Successive snapshots are taken after approximately equal intervals of time.

Kinetic Energy of an Ion in a Magnetic Bottle

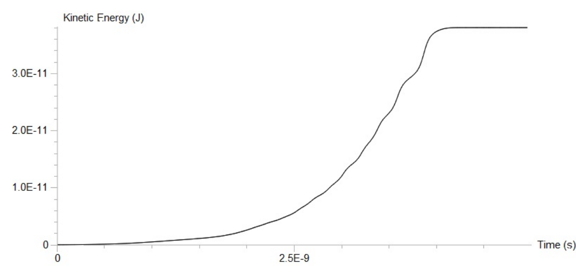

Figure 7: The graph of the kinetic energy vs. time of the ion in Figure 6.

Conclusions

The magnetic field of a magnetic bottle is strongest near either coil and weakest near the center. The field is nearly uniform near the center, and it begins to curve away from the bottle along the circumference of the coils.

In order to be trapped, an ion should be directed toward the center of the magnetic bottle and pass through either coil. This ensures that that there will be a maximal ratio of strong field strength to weak field strength. Once inside the bottle, only an ion obeying equation 8,

$$\frac{|v_\perp|}{|v_\parallel|} \geq \frac{1}{\sqrt{R_m}},$$

will be trapped. Thus, there are a large number of ion speeds that can be trapped by a magnetic bottle, provided that the ion's velocity obeys this equation. An ion that does not fit this criteria escapes the bottle, and is said to be in the loss cone.

For a trapped ion, it was found that the polarity of its charge, the direction of its perpendicular velocity, and the coil through which it first entered the bottle does not change whether it was trapped. The magnitude of its charge, however, does have an effect, though no quantitative description of this was found.

The kinetic energy of a trapped ion was found to increase during the time that it was contained in the bottle. Because of this, its speed will increase and it eventually will not have the required velocity criteria, and so will escape the bottle.

vPython Code

Calculating the Magnetic Field

```
from __future__ import division
from visual import *
from visual.graph import *

#CONSTANTS
mu0=2*pi*1e-7
    #RING PARAMATERS
rad=0.02 #meters, radius
rad1=rad
rad2=rad
I=10 #Amps
I1=I
I2=I
distance=0.05 #meters, distance between coils
    #STEPS, INCREMENTS, and SCALE FACTORS
ystep=distance/10
xstep=ystep
zstep=ystep
scale=2*pi/50 #Determines how small dl is along coil

#SCENE
scene=display(title='Magnetic Field Lines of Magnetic Bottle',
              background=color.white)

#CREATING COILS
coil1=ring(pos=(0,-distance/2,0),radius=rad,axis=(0,1,0),material=materials.blazed)
coil2=ring(pos=(0,distance/2,0),radius=rad,axis=(0,1,0),material=materials.blazed)

#CALCULATING MAGNETIC FIELD
for y in arange(-0.75*distance,0.75*distance,1.8*ystep):
    for x in arange(-1.2*rad,1.2*rad,xstep):
        for z in arange(-1.2*rad,1.2*rad,xstep):
            B=vector(0,0,0) #Initializing Bfield for new point

            for t in arange(0,2*pi+scale,scale):
                #Position vector of small current segment
                dl1pos=vector(rad1*cos(t),-distance/2,rad1*sin(t))
                dl2pos=vector(rad2*cos(t),distance/2,rad2*sin(t))
                #Tangent vector of small current segment
                dl1=-1*rad1*scale*vector(-sin(t),0,cos(t)) #Sign switches for cusp
                dl2=rad2*scale*vector(-sin(t),0,cos(t))
                #Finding R, the distance from source point to field point
                R1=vector(x,y,z)-dl1pos
                R2=vector(x,y,z)-dl2pos

                #CALCULATING dB and B
                dB1=mu0*I1/(4*pi)*cross(dl1,R1)/(mag2(R1))
                dB2=mu0*I2/(4*pi)*cross(dl2,R2)/(mag2(R2))
                B=B+dB1+dB2

                #PLOTTING Bfield
                field=arrow(pos=vector(x,y,z),axis=norm(B),length=1800*mag(B),
                            color=color.red)
```

Ion Trajectories

```
from __future__ import division
from visual import *
from visual.graph import *

#CONSTANTS
mu0=2*pi*1e-7
e_mass=9.109e-31 #kg
Hmass=1822*e_mass #kg
e_charge=1.6e-19 #C
c=3e8 #m/s, speed of light in vacuum
    #RING PARAMATERS
turns=1*700000000
rad=0.01 #meters, radius
rad1=rad
rad2=rad
I=1000 #Amps
I1=I
I2=I
distance=0.025 #meters, distance between coils
    #ION PARAMATERS
ionrad=distance/50
ionpos=vector(-1/3*rad,0.51*distance,0)
ionvel=vector(5e6,-5e6,0)
ionmass=Hmass
ioncharge=1*e_charge
ionmom=ionmass*ionvel
    #STEPS, INCREMENTS, and SCALE FACTORS
ystep=distance/10
```

```
xstep=ystep
zstep=ystep
scale=2*pi/50 #Determines how small dl is along coil
    #TIME
t=0
dt=1e-12
tmax=0.01

scene.background=color.white

#CREATING CHARGED PARTICLE
ion=sphere(pos=ionpos,radius=ionrad,charge=ioncharge,velocity=ionvel,
           momentum=ionmom,color=color.blue,make_trail=True,retain=800)


#CREATING COILS
coil1=ring(pos=(0,-distance/2,0),radius=rad,axis=(0,1,0),material=materials.blazed)
coil2=ring(pos=(0,distance/2,0),radius=rad,axis=(0,1,0),material=materials.blazed)

#KE GRAPH
KEgraph=gdisplay(x=500,background=color.white,foreground=color.black,
                 xtitle='Time (s)',ytitle='Kinetic Energy (J)')
KE=gcurve(gdisplay=KEgraph)

#MAIN LOOP
while (t<tmax):
    rate(100000000)

    B=vector(0,0,0) #Initialize for particle's new position

    #CALCULATING BFIELD AT PARTICLE'S POSITION
    for theta in arange(0,2*pi+scale,scale):
        #Position vector of small current segment
        dl1pos=vector(rad1*cos(theta),-distance/2,rad1*sin(theta))
        dl2pos=vector(rad2*cos(theta),distance/2,rad2*sin(theta))

        #Tangent vector of small current segment *I-DIRECTION SWITCHES FOR CUSP
        dl1=rad1*scale*vector(-sin(theta),0,cos(theta))
        dl2=rad2*scale*vector(-sin(theta),0,cos(theta))

        #Finding R, the distance from source point to field point
        R1=ion.pos-dl1pos
        R2=ion.pos-dl2pos

        #CALCULATING dB and B
        dB1=mu0*I1/(4*pi)*cross(dl1,R1)/(mag2(R1))
        dB2=mu0*I2/(4*pi)*cross(dl2,R2)/(mag2(R2))
        B=B+dB1+dB2

    #UPDATING FORCE, MOMENTUM, VELOCITY, and POSITION
    F_B=cross(ioncharge*ionvel,turns*B/400)
    ionmom=ionmom+F_B*dt
    ionvel=ionmom/ionmass
    ion.pos=ion.pos+ionvel*dt
    t=t+dt

    KE.plot(pos=(t,1/2*ionmass*(mag(ionvel)**2)))
```