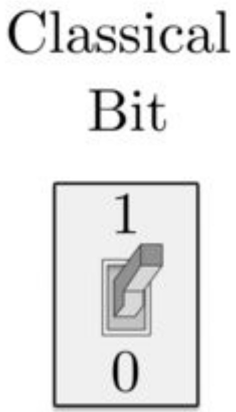
QuIC Seminar - Introduction and Math Review

Joe Kitzman

Classical Information Processing



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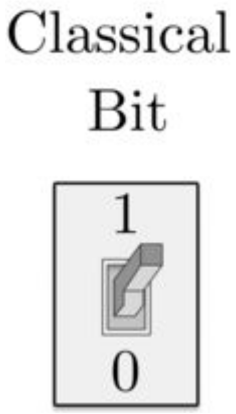
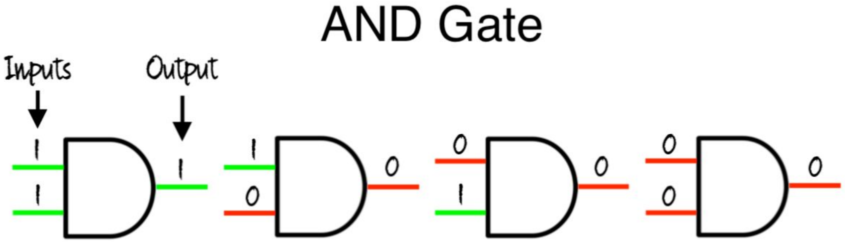
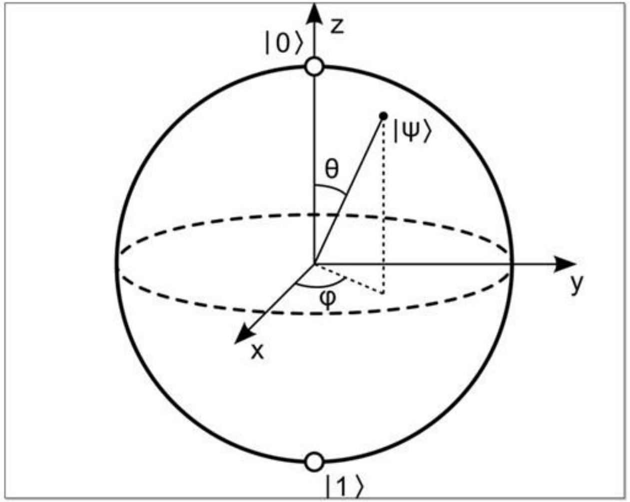




Quantum Information Processing



$$|\psi\rangle = a\,|0\rangle + b\,|1\rangle \qquad \psi = \begin{pmatrix} a \\ b \end{pmatrix}$$

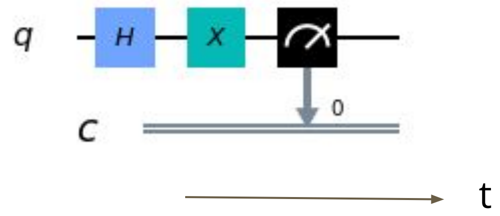
$$\langle\psi|\psi\rangle = \left(\langle 0|\, a^* + \langle 1|\, b^*\right)\left(a\,|0\rangle + b\,|1\rangle\right) = 1$$

$$|a|^2 + |b|^2 = 1$$

Gate Model of Quantum Computing

$$i\hbar \frac{\partial |\psi(t)\rangle}{\partial t} = \hat{H} |\psi(t)\rangle$$

$$|\psi(t)\rangle = e^{\frac{-i\hat{H}t}{\hbar}} |\psi(0)\rangle = \hat{U}(t) |\psi(0)\rangle$$



Observables and Measurement

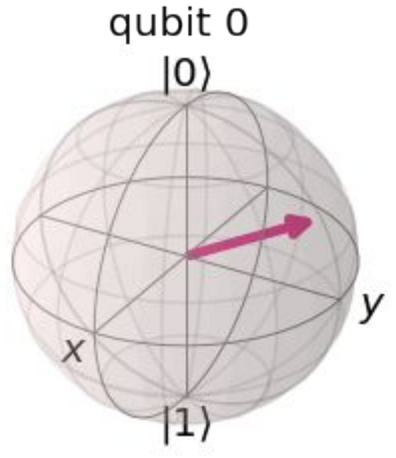
In physics we measure physical properties of the system, we call these *observables*.

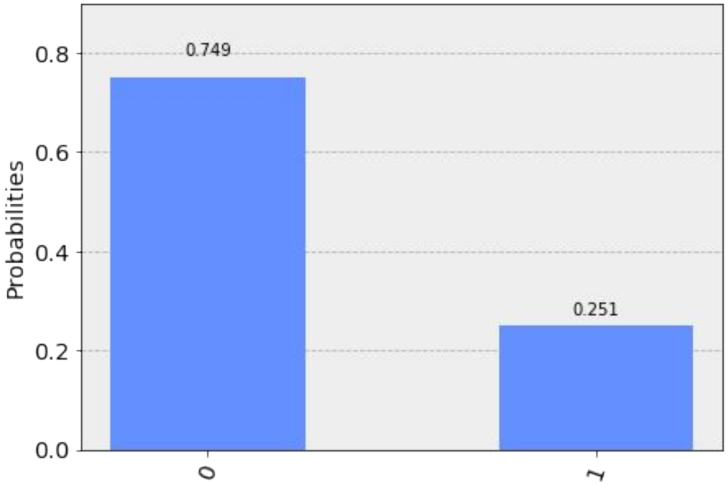
$$\hat{A} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \hat{A} \rightarrow \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Additionally, observables are Hermitian: $\hat{A}^\dagger = \hat{A}$

Observables and Measurement

Measurement in quantum mechanics is *destructive*





Observables and Measurement

Let's look at a single qubit example:

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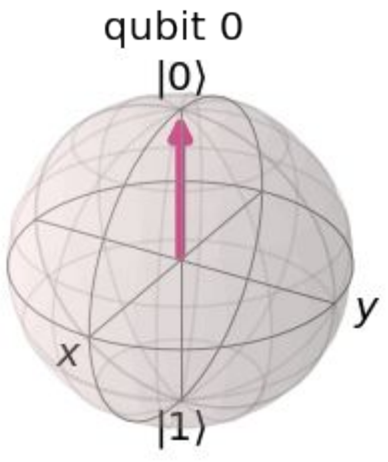
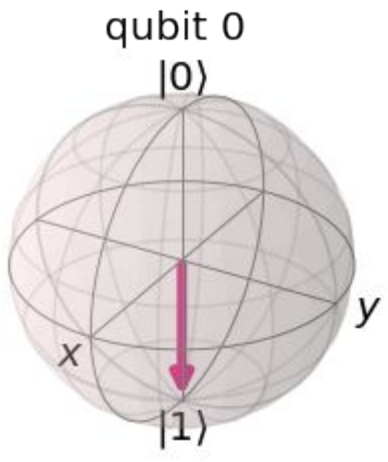
$$\langle\psi|a_i\rangle\langle a_i|\psi\rangle = |\langle a_i|\psi\rangle|^2$$

$$(\langle 0|a^* + \langle 1|b^*)|0\rangle\langle 0|(a|0\rangle + b|1\rangle) = |a|^2\langle 0|0\rangle^2 = |a|^2$$

Introduction to Quantum Gates

Consider an operator which acts on the basis states as follows:





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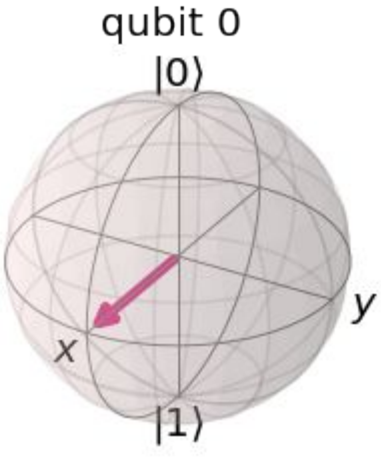
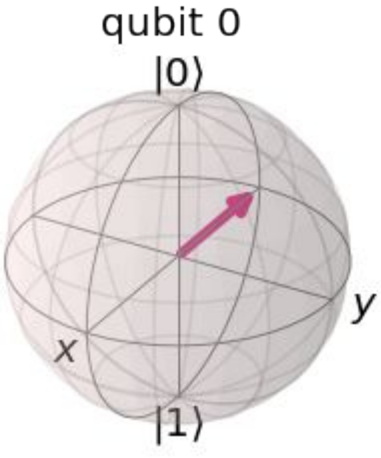


$$|{+x}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$



$$|{-x}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

Introduction to Quantum Gates

One-Qubit Hadamard Gate

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\hat{H}|0\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$\hat{H}|1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

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Question: What happens if I apply two Hadamard gates to a basis vector? What does this say about the inverse of the Hadamard gate?

Important Property of Single Qubit Gates

Single qubit gates are *unitary* - what does this mean?

$$\hat{U}^\dagger \hat{U} = \hat{I}$$

$$|\psi'\rangle = \hat{U}\,|\psi\rangle$$

$$\langle\psi'|\psi'\rangle = \langle\psi|\,\hat{U}^\dagger \hat{U}\,|\psi\rangle = \langle\psi|\psi\rangle$$

More Quantum Gates

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

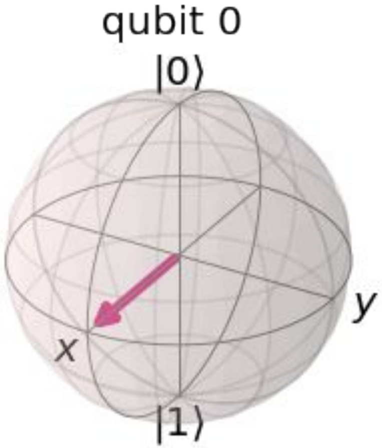
More Quantum Gates

$\sigma_x =$

$\sigma_z =$

Question: What do the Pauli gates represent in terms of a rotation on the Bloch Sphere?

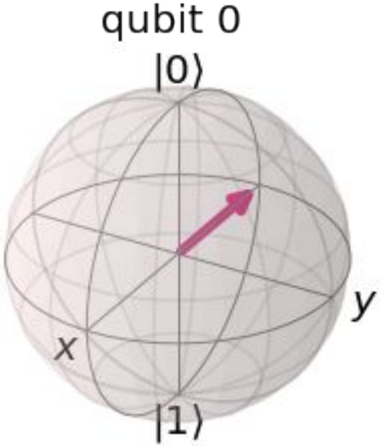
More Quantum Gates



$$|+x\rangle$$

$$\sigma_z \, |+x\rangle =$$

More Quantum Gates



Multi-Qubit Gates and Entanglement

So far - only single qubit gates: what if I add more qubits to my system?

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$$|\psi\rangle = \psi_0 |0\rangle + \psi_1 |1\rangle$$

$$|\phi\rangle = \phi_0 |0\rangle + \phi_1 |1\rangle$$

$$|\psi\rangle \otimes |\phi\rangle = (\psi_0 |0\rangle + \psi_1 |1\rangle) \otimes (\phi_0 |0\rangle + \phi_1 |1\rangle)$$

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$$|\psi\rangle \otimes |\phi\rangle = \psi_0\phi_0 |0\rangle \otimes |0\rangle + \psi_1\phi_0 |1\rangle \otimes |0\rangle + \psi_0\phi_1 |0\rangle \otimes |1\rangle + \psi_1\phi_1 |1\rangle \otimes |1\rangle$$

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$$|\psi\rangle \otimes |\phi\rangle = \psi_0\phi_0 |00\rangle + \psi_1\phi_0 |10\rangle + \psi_0\phi_1 |01\rangle + \psi_1\phi_1 |11\rangle$$

Multi-Qubit Gates and Entanglement

A state is called *entangled* if it cannot be written as a tensor product of states:

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Not entangled!

$$\frac{|01\rangle + |00\rangle}{\sqrt{2}} = |0\rangle \otimes \frac{|1\rangle + |0\rangle}{\sqrt{2}}$$

Entangled!

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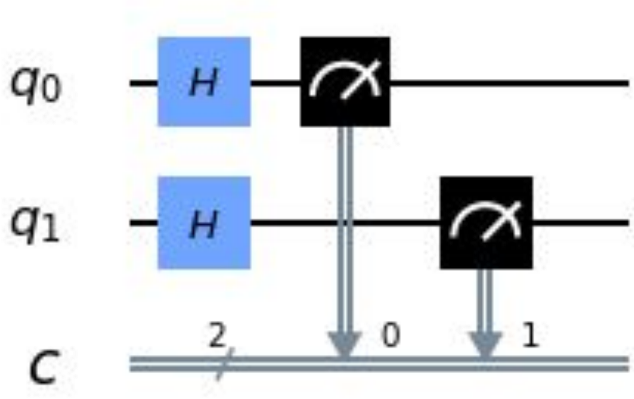
Multi-Qubit Gates and Entanglement

How do I write operators for multi-qubit states?

Question: How can I write an operation on one qubit while leaving the other qubit alone in this context?

Quantum Circuit for a 2 Qubit System



The CNOT Gate

In words - if the first (control) qubit is in the |1> state, flip the second (target) qubit. If not, do nothing.

This tells us how the basis vectors transform:

$$\hat{C}\,|10\rangle = |11\rangle$$

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$$\hat{C}\,|01\rangle = |01\rangle$$

$$\hat{C} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

The CNOT Gate

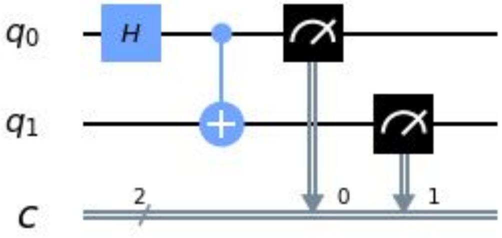
Important note -

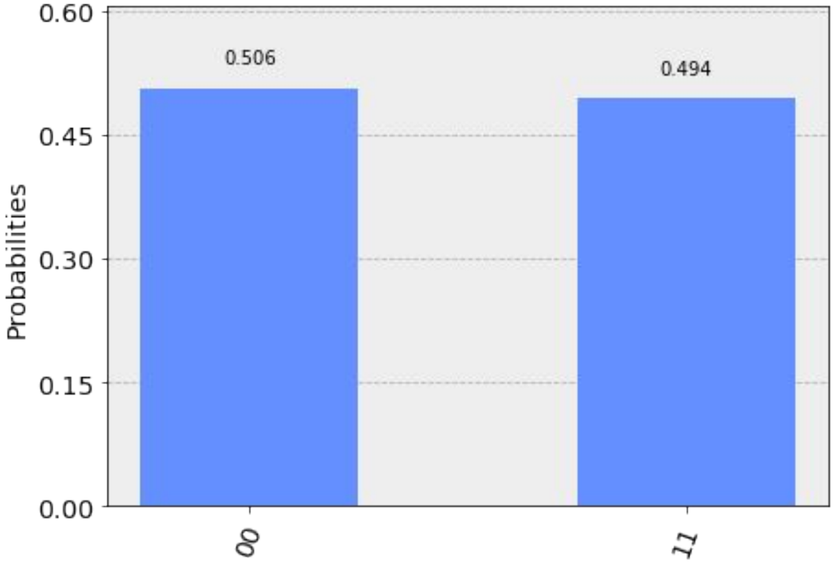
The CNOT gate *cannot* be written as an outer product of two single qubit gates

This means that we can entangle two qubits using this (and more) gates!

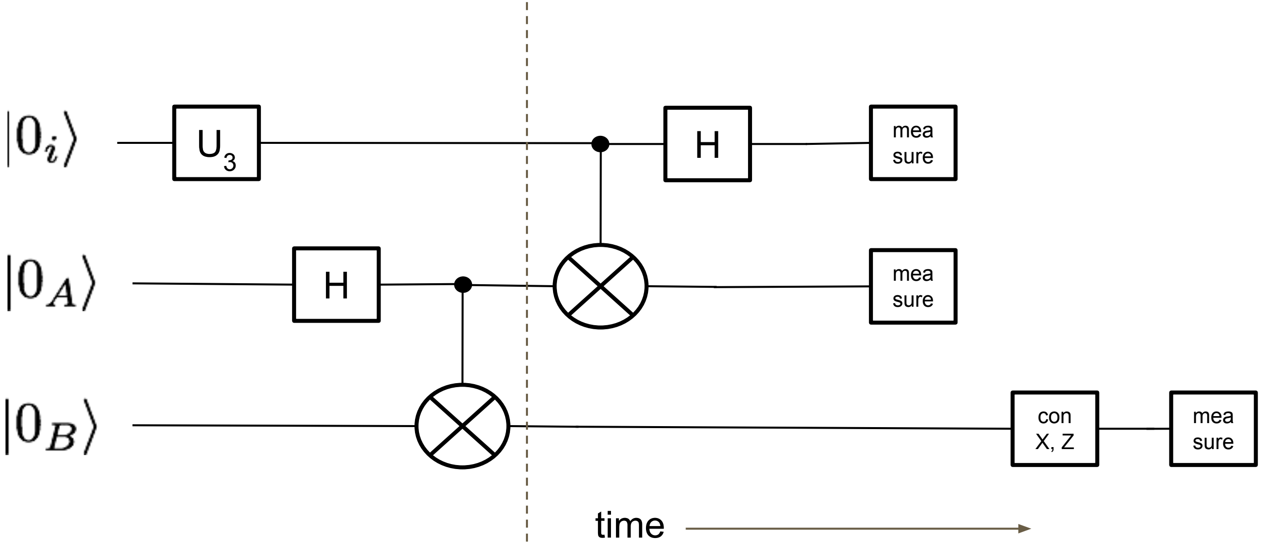
$$|00\rangle \rightarrow |0\rangle \otimes H|0\rangle = \frac{|00\rangle + |01\rangle}{\sqrt{2}} \rightarrow CNOT_{01}\frac{|00\rangle + |01\rangle}{\sqrt{2}} = \frac{CNOT_{01}|00\rangle + CNOT_{01}|01\rangle}{\sqrt{2}} = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

The CNOT Gate





Quantum Teleportation



The Density Matrix

Question: How can I write a single particle wavefunction for *one* of the qubits the Bell (or any other entangled) state??

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Answer: You can't! We need a new tool to characterize these systems - the density matrix!

The Density Matrix

$$\rho = |\psi\rangle \langle\psi|$$

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Some important properties:

$$Tr(\rho) = 1$$

$$\rho^\dagger = \rho$$

$$\langle\hat{A}\rangle = Tr(\rho\hat{A})$$

Density Matrix Continued

A more concrete example:

$$|\psi\rangle \rightarrow \mathcal{H}_A$$

$$|\phi\rangle \rightarrow \mathcal{H}_B$$

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

$$\rho_{AB} = |\psi\varphi\rangle\langle\psi\varphi| = \rho_{11}|00\rangle\langle 00| + \rho_{12}|00\rangle\langle 01| + \rho_{13}|00\rangle\langle 10| + \ldots + \rho_{44}|11\rangle\langle 11| = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix}$$

Density Matrix for the Bell State

$$\rho = \frac{1}{2}\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right)$$

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Note: The choice of basis
is important here!

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

Density Matrix Continued

In order to find expectation values for *one* qubit (let's say the qubit that lives in H_A), we use something called a *partial trace* and trace over the degrees of freedom in H_B: block the density matrix up into submatrices:

$$\rho_A = \mathrm{Tr}_B(\rho_{AB}) = \begin{pmatrix} \mathrm{Tr}\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} & \mathrm{Tr}\begin{pmatrix} \rho_{13} & \rho_{14} \\ \rho_{23} & \rho_{24} \end{pmatrix} \\ \mathrm{Tr}\begin{pmatrix} \rho_{31} & \rho_{32} \\ \rho_{41} & \rho_{42} \end{pmatrix} & \mathrm{Tr}\begin{pmatrix} \rho_{33} & \rho_{34} \\ \rho_{43} & \rho_{44} \end{pmatrix} \end{pmatrix}$$

Density Matrix Continued

In order … that lives in
H_A), we … ees of
freedom …

Question: What do I do differently if I want to trace over the degrees of freedom of H_A?

ρ_A … ρ_{14} ρ_{24} ρ_{34} ρ_{44}

Density Matrix Continued

Question: Why is the partial trace so powerful? What does this allow me to do?

Density Matrix Continued

Question: Why is the partial trace so powerful? What does this allow me to do?

Answer: By tracing over degrees of freedom I don't care about (or may not have any information about) I can actually write down a smaller dimensional matrix describing what I do care about!

Pure vs Mixed States

$$\rho_A = \frac{\mathbf{I}}{2}$$

(Check this!)

What will give me:

$$|\psi\rangle \langle\psi| = \frac{\mathbf{I}}{2}$$

Pure vs Mixed States

$$\rho_A = \frac{\mathbf{I}}{2}$$

(Check this!)

No single wavefunction! I have "incomplete" information about the state! (Why?)

Pure vs Mixed States

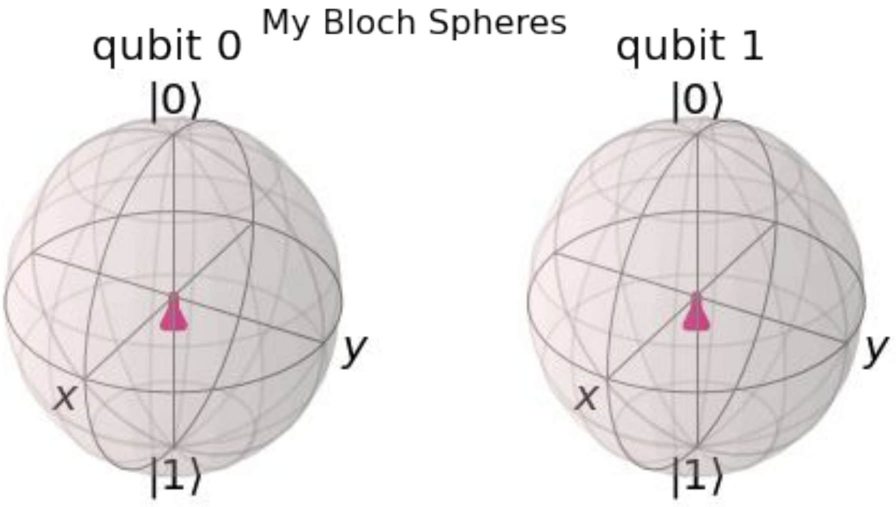
We need an ensemble of pure states!

Pure state: $\rho = |\psi\rangle \langle\psi|$

Mixed state: $\rho_A = \sum_A p_A |\psi_A\rangle \langle\psi_A|$

Property of pure states: $\rho^2 = \rho$

Pure vs Mixed States



Because of the partial trace, I only know about *probabilities* nothing about phases! This is called an *incoherent* superposition of |0> and |1>

Review - What Did We Learn?

Quantum bits (or qubits) exist in a superposition of classical states

Gates are used to manipulate the states of qubits

Certain gates can entangle qubits

Entangled qubits are best described via the *density matrix*

Resources

Helpful tool for visualization/qubit manipulation (qubits in Python!)

https://qiskit.org/textbook/preface.html (Chapters 1 and 2)

Quantum Computation and Quantum Information - Nielson and Chuang

(Ch 1 and 2)

Ask the organizers! They would be more than happy to talk quantum!

quic.seminar@gmail.com