QuIC seminar 11/26/18 and 12/3/18

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Abstract

For the better part of the last semester we've been talking about various algorithms that could be executed on a hypothetical quantum computer. The reason we're talking about any of this is because the actual act of building such a hypothetical computer has very recently been brought out of the realm of science fiction and into the realm of potential possibility. That being said, *it's still a really hard problem.* This lecture isn't intended to be a master course on how to build a quantum computer (nobody knows how to do that yet), but rather an introduction to *why* it's such a hard problem, and hopefully an introduction to why a lot of very smart people believe that, even though the problem is hard, it's tractable. The discussion of Rabi oscillations roughly follows [1], while the discussion of quantum error correction more closely follows (copies) a section of [3], which is a phenomenal set of lecture notes I suggest you read.

1 Why do classical computers work?

To start to understand the pitfalls in actually implementing a quantum computer, it's useful to think of how we got *classical* computers to work so well. After all, classical computers have become completely ubiquitous in modern life: we almost all carry at least one on us at any point in the day, and have dozens that help of do almost everything. How did classical computers become such an unrelenting force in our lives?

Classical computers are built on digital logic circuits. As you probably know, digital logic circuits take in some binary input (0 or 1, yes or no, high or low voltage, etc...) and feed these binary inputs through a series of fundamental logic gates, which are the fundamental building blocks of the algorithms executed. Physically, the binary logic states take the form of high and low voltages on a wire, and the logic gates are build out of metal oxide semiconductor field effect transistors (MOSFETs). Why does this work?

1. It's relatively easy to build all the logic gates we need out of MOSFETs
2. MOSFETs are fast.
3. MOSFETs are miniaturizable and scalable. A modern iPhone has on the order 10^9 MOSFETs in it's processor.
4. MOSFETs are pretty robust to error.
5. When a error does occur, *it's easy to detect and correct.*

This list is, of course, oversimplified, but it serves as a good starting point for what we might want out of a quantum computer. With that in mind, let's investigate a generalized circuit, and think about the physical assumptions we've been making when writing these circuits down.

2 What goes into a quantum circuit?

For the better part of the past semester, we've been writing down circuits that look something like this:

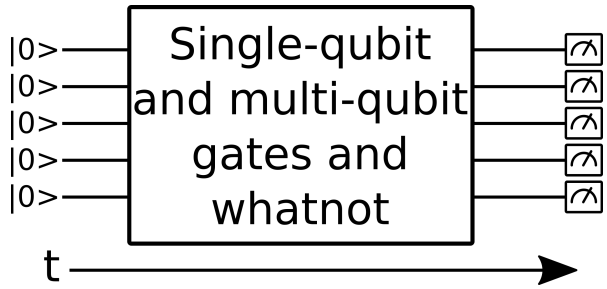

Figure 1: That's an algorithm alright

What assumptions have we been making here?

1. First off, qubits are physical quantum two-level systems. We should have some number of quantum two level systems to work with.
2. at $t = 0$, all of our qubits are in $|0\rangle$. Physically, this means that we had better be able to initialize all of our qubits to some uniform state (this tends to be the ground state, i.e. the state of lower energy) at the beginning of our computation.
3. At the end of our algorithm, we always measure the qubit, projecting it into $|0\rangle$ or $|1\rangle$ to determine the result of of computation. We had better be able to measure the state of our qubits, and do so with a high fidelity.
4. Time moves forward as the we go left to right on our circuit. We know from the Shrödinger equation that the time evolution of each qubit will be governed by it's Hamiltonian. If we want to control each qubit individually, we had better be able to control the Hamiltonian of individual qubits in a manner to execute the single- and multi-qubit gates we need to do our computation.
5. Time is a precious resource for us: we had better be able to execute our gates quickly.
6. We will never have perfect control over the Hamiltonian of a quantum system, which means that there will be errors. We had best find a way to minimize these errors, and a way to correct the errors we can't get rid of.

We'll start off by discussing the qubit Hamiltonian today, since all of this is a nonstarter if we don't have sufficient control over the time evolution of our qubits. Afterwords, we'll move on to a brief discussion of quantum error correction.

3 The qubit Hamiltonian: getting to single qubit gates

From the time dependent Shrödinger equation, the time evolution of a quantum system is given by the unitary operator

$$U(t) = e^{-i\mathbf{H}t/\hbar} \tag{1}$$

where $\mathbf{H}$ is the Hamiltonian of the system. For a single quantum two level system, the Hamiltonian is given by.

$$\mathbf{H_{bare}} = -\frac{\hbar\omega_q}{2}\sigma_z \tag{2}$$

Where σ_z is the Pauli-Z operator. Of course, our Hamiltonian is an operator that acts on a two level system and brings it to another two level system: it should be a 2 x 2 matrix.

Derailing the story a bit, we already have an important requirement for to satisfy point (2): a quantum two level system has two energy levels, which are spaced by $\hbar\omega_q$. The physicists in the room know that, at any finite temperature, there's always a bath of thermal photons with some distribution that peaks at $\hbar\omega_\gamma \sim K_b T$. If any of these photons have energy $\hbar\omega_\gamma = \hbar\omega_q$, they can randomly excite the qubit from $|0\rangle$ to $|1\rangle$. So, to satisfy point (2), we need to have $K_b T << \hbar\omega_q$. This is why a lot of work on quantum computing happens at dilution refrigerator temperatures (≤ 50 mK).

Anyways, the time evolution operator associated with this Hamiltonian is $U(t) = e^{i\omega_q t\sigma_z/2}$. This isn't very convenient: it's as if we were always applying a Pauli Z gate to our qubit. What we'd really like is a Hamiltonian that looks something like

$$\mathbf{H} = X(t)\sigma_x + Y(t)\sigma_y + Z(t)\sigma_z \tag{3}$$

Where $X(t), Y(x)$, and $Z(t)$ are step functions that we turn on and off when we want to perform a gate.

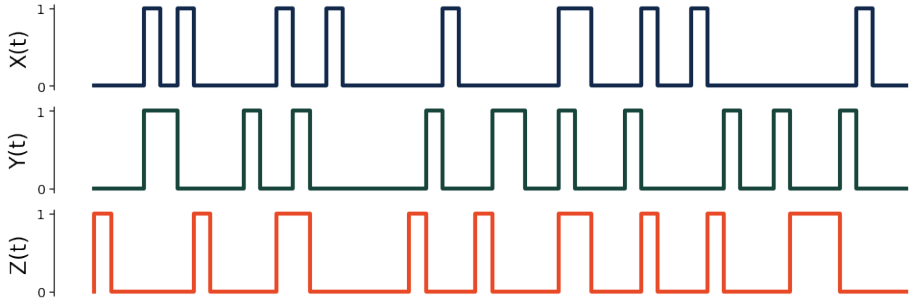

Figure 2: The hamiltonian should look something like this.

For example, say $Z(t) = \hbar A$ when it is "on", and $Z(t) = 0$ otherwise. If we wanted to perform a Pauli Z gate, which is described by the unitary operator[1] $e^{-i\frac{\pi}{2}\sigma_z}$, we would just turn $Z(t)$ "on" for time $\tau = \pi/2A$.

In general, the way we go about manipulating the qubit Hamiltonian is by using coherent electromagnetic radiation to drive the qubit in a deterministic fashion. What follows is a semi-classical treatment of the famed Rabi oscillation problem which describes the interaction of a quantum two level system with a classical coherent electric field.

3.1 Rabi Oscillations

We begin by assume that our qubit is much smaller than the wavelength of whatever radiation interacts with it[2]. This allows us to make the dipole approximation for an atom interacting with a time dependent (classical) electric field:

$$\mathbf{H} \approx \mathbf{H}_{bare} - \vec{\mathbf{d}} \cdot \vec{E}(t) \tag{4}$$

where $\vec{\mathbf{d}}$ is the dipole operator.

Recall the important identity that, for a complete basis of states $|i\rangle$, $\mathbf{I} = \sum_i |i\rangle\langle i|$, where I is the identity. Since for a qubit, $|0\rangle$ and $|1\rangle$ define a complete basis of states, we can insert the identity and write our Hamiltonian as

[1] up to an overall phase factor

[2] this is generally true: even superconducting qubits, which can be milimeter scale, generally are engineered to have transition frequencies of 5-10 GHz, at which light has a wavelength of order several cm.

$$
\begin{aligned}
&-\frac{\hbar\omega_q}{2}\sigma_z - (|0\rangle\langle 0| + |1\rangle\langle 1|)\vec{\mathbf{d}}\cdot\vec{E}(t)(|0\rangle\langle 0| + |1\rangle\langle 1|) \\
&-\frac{\hbar\omega_q}{2}\sigma_z - \langle 0|\vec{\mathbf{d}}\cdot\vec{E}(t)|1\rangle|0\rangle\langle 1| + \langle 1|\vec{\mathbf{d}}\cdot\vec{E}(t)|0\rangle|1\rangle\langle 0| \\
&-\frac{\hbar\omega_q}{2}\sigma_z - \langle 0|\vec{\mathbf{d}}\cdot\vec{E}(t)|1\rangle\sigma_x
\end{aligned} \tag{5}
$$

Where I've used parity selection rules to constrain which matrix elements are 0 ($\vec{\mathbf{d}}\cdot\vec{E}(t)$ is negative under parity) .

Now, for the sake of simplicity we will constrain ourselves to electric fields of the form $\vec{E}(t) = \vec{E}_0\cos\omega t$, with $|\omega - \omega_q| << \omega_q$. If we write $\langle 0|\vec{\mathbf{d}}\cdot\vec{E}_0|1\rangle = A$, we arrive at the Hamiltonian

$$
\mathbf{H}_{Rabi} = -\frac{\hbar\omega_q}{2}\sigma_z - A\cos\omega t\sigma_x \tag{6}
$$

We'll now set up the physical situation we want to describe: At $t = 0$ our qubit is sitting happily in it's ground state, when all of a sudden we blast it with coherent electromagnetic radiation at a frequency near its transition frequency. What does our qubit do, as a function of time?

In accordance with time dependent perturbation theory, we'll assume it's safe to write the wave function as

$$
|\Psi(t)\rangle = C_0(t)e^{-iE_0t/\hbar}|0\rangle + C_1(t)e^{-iE_1t/\hbar}|1\rangle \tag{7}
$$

Where our goal is to solve for C_0 and C_1, the (time dependent) probabilities of being in the ground/excited state respectively. Here, $E_0 = -\hbar\omega_q/2$ and $E_1 = +\hbar\omega_q/2$ are the unperturbed energies of Eqn (2). We then plug our guess for the wave function into the time dependent Schrödinger equation, we get a set of coupled differential equations:

$$
\dot{C}_0 = \frac{i}{\hbar}A\cos(\omega t)e^{-i\omega_q t}C_1 \tag{8a}
$$

$$
\dot{C}_1 = \frac{i}{\hbar}A\cos(\omega t)e^{i\omega_q t}C_0 \tag{8b}
$$

If we expand $\cos(\omega t) = \frac{1}{2}(e^{i\omega t} + e^{-i\omega t})$, we see that each differential equation contains two terms (modulo a minus sign): one containing a sum of frequencies $e^{i(\omega+\omega_q)t}$ and one a difference of frequencies $e^{i(\omega-\omega_q)t}$. We now invoke the **rotating wave approximation** (RWA): any dynamics (rotations) caused by the $e^{i(\omega+\omega_q)t}$ term will happen so quickly that, over time scales we care about, they will average out to zero. The RWA is valid in the regime $A/\hbar, |\omega_q - \omega| << \omega_q$, i.e. the drive amplitude and qubit/drive detuning are small compared to the qubit transition frequency. In this regime, since the time rate of change of the probabilities is set by the drive amplitude, the contribution from the "counter-rotating" term (the $e^{i(\omega+\omega_q)t}$ term) will oscillate from positive to negative rapidly the timescale at which the probabilities actually change. Thus, contributions from the counter-rotating term average out to zero, and we're free to drop it.

Dropping the counter-rotating terms, our differential equations become

$$
\dot{C}_0 = \frac{i}{2\hbar}Ae^{-i\Delta t}C_1 \tag{9a}
$$

$$
\dot{C}_1 = \frac{i}{2\hbar}Ae^{i\Delta t}C_0 \tag{9b}
$$

where I've introduced the detuning $\Delta = (\omega - \omega_q)$. If we now take the time derivative of the second line and plug it into the first, we can eliminate C_0 and reduce our problem down to the simple differential equation

$$\ddot{C_1} + i\Delta\dot{C_1} + \frac{A^2}{4\hbar^2}C_1 = 0 \tag{10}$$

Which has the general solution

$$C_1(t) = C_+ e^{i\lambda_+ t} + C_- e^{i\lambda_- t} \tag{11}$$

Where

$$\lambda_\pm = \frac{1}{2}\Big(\Delta \pm \sqrt{\Delta^2 + A^2/\hbar^2}\ \Big) = \frac{1}{2}(\Delta + \Omega_R) \tag{12}$$

where we've condensed the square root into one variable with units frequency: Ω_R, the Rabi frequency. Determining our boundary conditions by setting the qubit to the ground state at $t = 0$, we can, with a little bit of algebra, show

$$C_1(t) = \frac{A}{\hbar\Omega_R} e^{i\Delta t/2} \sin(\Omega_R t/2), \tag{13}$$

$$P_1(t) = |C_1(t)|^2 = \frac{A^2}{(\hbar\Omega_R)^2} \sin^2(\Omega_R t/2), \tag{14}$$

i.e., the qubit oscillates between the ground and the excited state at frequency $\Omega_R/2$. Plotted below is a theoretical 2D plot of the qubit excited state probability as both a function of time and drive/qubit detuning. You see a characteristic 'chevron' shape: as you detune the qubit and the drive, the oscillation frequency goes up, but the amplitude of the oscillations goes down.

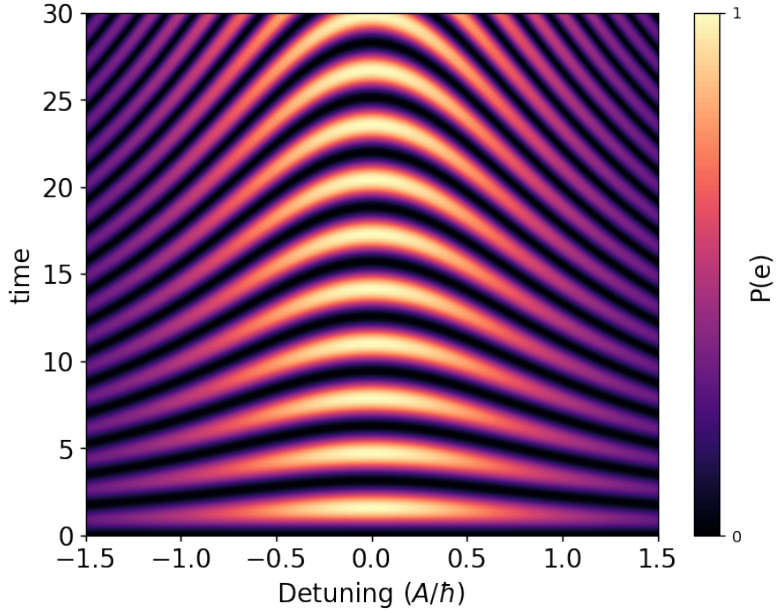

Figure 3: The Rabi 'chevron'

Thus, we've come to an important conclusion: by applying coherent radiation close to the transition frequency of a qubit with a dipole coupling we can change the state of the qubit from $|0\rangle$ to $|1\rangle$, or back again. Of course, the Rabi Hamiltonian contains σ_x, so in reality when we irradiate our qubit with a classical electromagnetic field, what we're really doing is "turning on $X(t)$" like we discussed earlier. To physically execute a Pauli-X (bitflip) gate, all we need to do is irradiate our qubit at its resonant frequency ($\Delta = 0$) for time $\tau = \pi\hbar/A$. As an exercise, convince yourself that that last sentence is true.

3.2 The Rotating Frame

We can shed even more light onto the problem by making a change of basis into a frame rotating along with the qubit state vector, called to rotating frame. We change bases by transforming our state by the unitary operator

$$\mathbf{R} = e^{-i\frac{\omega_q}{2}\sigma_z t} \tag{15}$$

If we apply this operator to the time dependent Schrödinger equation, we can calculate the effective hamiltonian in the rotating frame:

$$\begin{aligned}
\mathbf{R}\ i\hbar\frac{\partial}{\partial t}|\psi\rangle &= \mathbf{R}\mathbf{H}|\psi\rangle \\
i\hbar\Big(\frac{\partial}{\partial t}(\mathbf{R}|\psi\rangle) - \frac{\partial \mathbf{R}}{\partial t}|\psi\rangle\Big) &= \mathbf{R}\mathbf{H}\mathbf{R}^\dagger\mathbf{R}|\psi\rangle \\
i\hbar\frac{\partial}{\partial t}\mathbf{R}|\psi\rangle &= \Big[\mathbf{R}\mathbf{H}\mathbf{R}^\dagger + i\hbar\frac{\partial \mathbf{R}}{\partial t}\mathbf{R}^\dagger\Big]\mathbf{R}|\psi\rangle \\
\tilde{\mathbf{H}} &= \mathbf{R}\mathbf{H}\mathbf{R}^\dagger + i\hbar\frac{\partial \mathbf{R}}{\partial t}\mathbf{R}^\dagger
\end{aligned} \tag{16}$$

It is straightforward to check, in the rotating frame, our bare qubit Hamiltonian $\tilde{\mathbf{H}}_{bare} = 0$.

We'll write consider a very similar problem: a qubit dipole coupled to classical, coherent electromagnetic radiation. For this analysis, however, we'll do two things:

1. We'll assume that our electromagnetic radiation oscillates at at exactly $\omega = \omega_q$
2. We'll introduce a phase factor δ to our electric field

$$\mathbf{H}_{Rabi} = -\frac{\hbar\omega_q}{2}\sigma_z - A\cos(\omega_q t + \delta)\sigma_x \tag{17}$$

Transforming this hamiltonian to the rotating frame is kind of a bear (I did it in the supplemental materials), but the punchline is that, in the rotating frame, the Rabi Hamiltonian becomes

$$\tilde{\mathbf{H}}_{Rabi} = -\frac{A}{2}\big(\sigma_x\cos(\delta) + \sigma_y\sin(\delta)\big) \tag{18}$$

In other words *we can control which axis of the Bloch sphere we rotate about simply by changing the phase of the applied pulse.* We're one step closer to obtaining our theoretical Hamiltonian Eqn (3). The exact way to control the σ_z term of the Hamiltonian of a quantum two level system generally depends on the exact details of the system in question, so we won't discuss it here. However, since $Z = -iXY$, we already have a set of universal (if not very convenient) single qubit gates.

3.3 The Catch-22 of Quantum Computing

When we solved the Rabi oscillation problem, we specified that a Pauli-X gate can be executed by irradiating our qubit with radiation for time $\tau = \pi\hbar/A$. In effect, the speed of our quantum computer is then set by the magnitude of dipole coupling term $A = \langle 0|\vec{\mathbf{d}}\cdot\vec{E}_0|1\rangle$: the larger A is, the faster we can execute single qubit gates, and the faster our computer can run.

Now, for those of you who have taken a course or two in quantum mechanics, you are probably familiar with Fermi's Golden Rule

$$W_{if} = \frac{2\pi}{\hbar}|\langle f|\mathbf{H}'|i\rangle|^2\rho(E_i - E_f) \tag{19}$$

Fermi's golden rule states that the decay rate W_{if} of quantum state $|i\rangle$ to state $|f\rangle$ is proportional to the square of matrix element connecting the two states $|\langle f|\mathbf{H}'|i\rangle|$ and the density

of states at energy $E_i - E_f$ ($\rho(E_i - E_f)$) that the system can decay into. Put simply, if the the connection between $|i\rangle$ and $|f\rangle$ is strong, and there are a lot of ways for energy to escape into the environment, the state $|i\rangle$ will quickly decay into $|f\rangle$. If we want to use $|i\rangle$ and $|f\rangle$ as the two bases of a qubit, randomly decaying from $|i\rangle$ to $|f\rangle$ cause us to lose the information encoded in the qubit and introduce errors into our computation.

Inspecting the general Hamiltonian for a two level system dipole coupled to an electromagnetic field (Eqn. 5), the matrix element that connects the $|0\rangle$ and $|1\rangle$ states of the qubit we've been considering is the dipole coupling term $\langle 0|\vec{\mathbf{d}} \cdot \vec{E}(t)|1\rangle\sigma_x$, *the magnitude of which we've also determined sets the speed at which our quantum computer can operate.* We can do our best to make sure that $\vec{E}(t) = 0$ when we don't want to be performing a gate, but no matter what we do we can't get rid of all electromagnetic noise[3]. This is simply a more quantitative statement of the "catch-22 of quantum computing": in general, the same physical mechanisms that allow us to control quantum bits also cause the qubit to decohere. From this perspective, robust quantum error corrections is a necessity to do *any* meaningful computation on a quantum computer.

4 Into to quantum error correction

To get perspective on the problem of quantum error correction (QEC), it once again behooves us to consider correcting errors in classical information. For a classical, digital, computer, error correction is relatively straightforward:

1. Digital logic is robust to noise. Say I'm encoding my information in the voltages of wires in a circuit. If my zero state is $0V$ and my one is $5V$, I can be reasonably certain the $0.02V$ is zero and $4.87V$ is one.
2. Classical logic only has one type of error: the bit-flip error, where a one becomes a zero or vice versa.
3. I can always redundantly encode classical information. Say I have a some mechanism to physically encode one classical bit: over some time interval (say, one day) it has probability ϵ of flipping. We can simply redundantly encode this information into three **physical** bits, and call this group of 3 physical bits a **logical** bit. Whenever we want to access the information in this logical bit, we can read all three physical bits. If they're not all the same, we can simply use a majority vote to determine whether the bit is supposed to be 0 or 1. For example, lets say I encode a bit into my logical bit today, and check back in tomorrow. The probablility P_n of n physical bit-flips is

$$\begin{aligned} P_0 &= (1-\epsilon)^3 \\ P_1 &= 3\epsilon(1-\epsilon)^2 \\ P_2 &= 3\epsilon^2(1-\epsilon) \\ P_3 &= \epsilon^3 \end{aligned} \tag{20}$$

   If $\epsilon < 0.5$, it's easy to see that the error rate of our logical bit has a lower error rate than our physical bit.

If we try and apply these principals to quantum information, *none of them work*:

1. $\sqrt{0.99}|0\rangle + \sqrt{0.01}|1\rangle$ can't safely be assumed to be $|0\rangle$! When I measure this state, I'll record $|0\rangle$ 99% of the time, but 1% of the time I'll record $|1\rangle$ and be none the wiser that the bit was actually supposed to be $|0\rangle$. In this sense, quantum errors are *analog*, or continuous.

[3] Another strategy is to minimize the density of states at ω_q. This strategy turns out to be effective: see the superconducting "transmon" qubit [5]. This strategy can only get us so far: at some point, surface/substrate defects begin to dominate, and these loss sources are notoriously hard to control.

2. More than one type of error can occur. For example, in addition to the the bit-flip error discussed above, phase flip-errors can occur where the relative phase of the $|0\rangle$ and $|1\rangle$ components of the qubit state vector can change (for example, $1/\sqrt{2}(|0\rangle + |1\rangle)$ may change to $1/\sqrt{2}(|0\rangle - |1\rangle)$.) These errors are also continuous.
3. The no cloning theorem prevents me from copying qubits and using majority vote to determine which state the qubit is supposed to be in.
4. Even if I could clone my qubits, I couldn't use the majority voting game to correct errors, since measuring a qubit destroys any superpositions and projects it in to either $|0\rangle$ or $|1\rangle$.

Needless to say, the possibility of QEC looks pretty bleak. The answer to the problem, however, turns out to be contained in point 4: while state amplitudes/phases are continuous, *measurements are digital*. It turns out to be possible to construct clever QEC codes that take advantage of digital measurement and entanglement to correct errors.

4.1 A simple QEC code

While I'm not allowed to copy a quantum state, the transformation of three qubits

$$(\alpha|0\rangle + \beta|1\rangle) \otimes |00\rangle \Rightarrow \alpha|000\rangle + \beta|111\rangle \tag{21}$$

is unitary and legal (exercise: build a circuit to execute this.) We'll take this 3 qubit state to be our *logical* qubit. The single qubit operations for this state are

$$X_{log} = X_1 \otimes X_2 \otimes X_3 = X_1 X_2 X_3 \tag{22a}$$

$$Z_{log} = Z_1 Z_2 Z_3 \tag{22b}$$

$$Y_{log} = i X_{log} Z_{log} \tag{22c}$$

Where X_i is the Pauli-X operator acting on the i^{th} qubit, so on and so forth[4]. This means we're still at liberty to treat this group as a single qubit, albeit with our single qubit operators replaced with the logical operators defined above. Now, let's define some new operators called **stabilizers**

$$S_1 = Z_1 Z_2 \tag{23a}$$

$$S_2 = Z_2 Z_3 \tag{23b}$$

Clearly, $[S_1, S_2] = 0$ and $[S_1, Z_{log}] = [S_2, Z_{log}] = 0$. Surprisingly, using the identity $X_i Z_i = -Z_i X_i$ we can also show that

$$\begin{aligned} [S_1, X_{log}] &= Z_1 Z_2 X_1 X_2 X_3 - X_1 X_2 X_3 Z_1 Z_2 \\ &= (Z_1 X_1 Z_2 X_2 - X_1 Z_1 X_2 Z_2) X_3 \\ &= (Z_1 X_1 Z_2 X_2 - (-Z_1 X_1)(-Z_2 X_2)) X_3 \\ [S_1, X_{log}] &= 0 \end{aligned} \tag{24}$$

and, similarly, $[S_2, X_{log}] = 0$. By extension, $[S_1, Y_{log}] = [S_2, Y_{log}] = 0$, which means that, since the stabilizers commute with all three of our logical qubit operators, *measuring a stabilizer will not have any back action on our logical qubit.*: we can measure S_1 and S_2 and neither measurement will have any effect on the state of the logical qubit or the outcome of the other measurement.

[4]Note: I realize it's confusing to switch notation half way through the lecture notes, but the literature on QEC almost exclusively refers to the Pauli matricies as $X/Y/Z$, while most experimental literature still sticks with $\sigma_x/\sigma_y/\sigma_z$. Sorry.

To see how this works, let's say a bit-flip error occurred on the first physical qubit in our logical qubit: $\alpha|000\rangle + \beta|111\rangle \rightarrow \alpha|100\rangle + \beta|011\rangle$. This is equivalent to saying that we evolved our system by erroneously applying an X_1 operator to it. If I measure the first stabilizer S_1, I'll find

$$\begin{aligned}\langle S_1\rangle &= (\langle 100|\alpha^* + \langle 011|\beta^*)Z_1Z_2(\alpha|100\rangle + \beta|011\rangle) \\ &= \alpha^2\langle 100|Z_1Z_2|100\rangle + \alpha^*\beta\langle 100|Z_1Z_2|011\rangle + \beta^*\alpha\langle 011|Z_1Z_2|100\rangle + \beta^2\langle 011|Z_1Z_2|011\rangle \\ &= -\alpha^2 - \beta^2\end{aligned}$$

$$\langle S_1\rangle = -1 \tag{25}$$

With relative ease, I can also prove $\langle S_2\rangle = +1$. In fact, we can make a table of possible bit-flip errors and the expectation value of both stabilizers measuring a logical qubit corrupted by said bit-flip error:

Error Type	$\langle S_1\rangle$	$\langle S_2\rangle$
I (none)	1	1
X_1	-1	1
X_2	1	-1
X_3	-1	-1

Note that none of these measurements tells us *anything about α or β!* By measuring a stabilizer, I gain no information about the state, so I don't destroy it. However, from the result of my measurement, I can uniquely identify if a bit-flip error occurred, and on which qubit the error occurred, and am in a position to rectify the error.

Since this is quantum mechanics and everything is painful, we can also be in a superposition of an error and no error. For example, if we write $|\psi\rangle = \alpha|000\rangle + \beta|111\rangle$, our system can evolve to a state where there is some small probability amplitude ϵ that a bitflip has occurred

$$|\psi\rangle \rightarrow (\sqrt{1-\epsilon^2}I + \epsilon X_1)|\psi\rangle \tag{26}$$

This state is, however, not an eigenstate of our stabilizer S_1. If we measure S_1 it will project the logical qubit into one of two states: (1) with probability $1-\epsilon^2$ it will project into a state where no error has occurred, and (2) with probability ϵ^2 it will project into $X_1|\psi\rangle$. If option (1) happens, we're done. *The measurement has corrected the error for us!* If option (2) occurs, the stabilizer measurements tell us that a bit-flip error has occurred on qubit 1, and once again we're in a position to rectify the error. The measurement process, in effect, *turns analog errors into either digital errors or no errors at all!*

This is a simple code that corrects only one type of error. There are plenty of QEC codes that are vastly more complicated[5]. As error correction codes get more complicated, an important metric of interest emerges: how many physical qubits do I need to make one logical qubit? As of writing this, the answer to that question is effectively ∞: no logical qubit robust enough to, say, break RSA has been experimentally implemented. In fact, we've only recently gotten to the point where certain specialized error correcting codes can lengthen time before a physical qubit decoheres (see [4] as an example.) If this sounds depressing, that's the reality of quantum computing: it's a damn hard problem. Our generation will likely be tasked with figuring out how to make, as John Preskill puts it [6], Noise Intermediate Scale Quantum (NISQ) devices, and how to run useful computations on NISQ devices.

[5]The current gold standard of QEC codes within the experimental community a code called called the surface code, which uses a 2D array of qubits. A lengthy introduction to the surface code may be found in [2].

A Supplementary materials

A.1 Rabi Hamiltonian in the Rotating frame

In order to derive Eqn (18), we need to plug our Rabi Hamiltonian (17) into

$$\tilde{\mathbf{H}} = \mathbf{R}\mathbf{H}\mathbf{R}^\dagger + i\hbar\frac{\partial \mathbf{R}}{\partial t}\mathbf{R}^\dagger$$

Where $\mathbf{R} = e^{-i\frac{\omega_q}{2}\sigma_z t}$ The second term gives

$$i\hbar\frac{\partial \mathbf{R}}{\partial t}\mathbf{R}^\dagger = \frac{\hbar\omega_q\sigma_z}{2}\mathbf{R}\mathbf{R}^\dagger = \frac{\hbar\omega_q\sigma_z}{2}$$

While the transformation of the unperturbed Hamiltonian gives us

$$\mathbf{R}\mathbf{H_0}\mathbf{R}^\dagger = \mathbf{R}(-\frac{\hbar\omega_q\sigma_z}{2})\mathbf{R}^\dagger = \mathbf{R}\mathbf{R}^\dagger(-\frac{\hbar\omega_q\sigma_z}{2}) = -\frac{\hbar\omega_q\sigma_z}{2}$$

Since $[\mathbf{R}, \sigma_z] = 0$. There two terms clearly cancel, so all we need to worry about is the transformation of the perturbative Hamiltonian. This term is

$$\mathbf{R}\mathbf{H}'\mathbf{R} = -A\cos(\omega_q t - \delta)(e^{-i\omega_q t\sigma_z/2}\sigma_x e^{+i\omega_q t\sigma_z/2})$$

We can invoke a particularly useful version of the Baker-Campbell-Hausdorff formula:

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]]... \tag{27}$$

to write down

$$= -A\cos(\omega_q t - \delta)\Big(\sigma_x + (i\omega_q t/2)[\sigma_z, \sigma_x] + \frac{1}{2!}(i\omega_q t/2)^2[\sigma_z, [\sigma_z, \sigma_x]] + \frac{1}{3!}(i\omega_q t/2)^3[\sigma_z, [\sigma_z, [\sigma_z, \sigma_x]]]...\Big)$$
$$= -A\cos(\omega_q t - \delta)\Big(\sigma_x + \omega_q t\sigma_y - \frac{(\omega_q t)^2}{2!}\sigma_x - \frac{(\omega_q t)^3}{3!}\sigma_y...\Big)$$
$$= -A\cos(\omega_q t - \delta)(\cos(\omega_q t)\sigma_x + \sin(\omega_q t)\sigma_y)$$

Where I've used the fact that $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$. We can now switch over to matrix notation:

$$= -A\cos(\omega_q t - \delta)\begin{bmatrix} 0 & \cos(\omega_q t) - i\sin(\omega_q t) \\ \cos(\omega_q t) + i\sin(\omega_q t) & 0 \end{bmatrix}$$
$$= \frac{A}{2}(e^{i(\omega_q t - \delta)} + e^{-i(\omega_q t - \delta)})\begin{bmatrix} 0 & e^{-i\omega_q t} \\ e^{i\omega_q t} & 0 \end{bmatrix}$$
$$= \frac{A}{2}\left(\begin{bmatrix} 0 & e^{-i\delta} \\ e^{i(2\omega_q t - \delta)} & 0 \end{bmatrix} + \begin{bmatrix} 0 & e^{-i(2\omega_q t - \delta)} \\ e^{i\delta} & 0 \end{bmatrix}\right)$$

Once again we invoke the rotating wave approximation: the $2\omega_q t$ terms rotate so fast compared to the timescales at which the probability amplitudes change that we can drop them. We arrive at the final form of the Rabi Hamiltonian in the rotating frame

$$= \frac{A}{2}\begin{bmatrix} 0 & e^{-i\delta} \\ e^{i\delta} & 0 \end{bmatrix} = -\frac{A}{2}(\cos(\delta)\sigma_x + \sin(\delta)\sigma_y) \tag{28}$$

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