# Variational Quantum Linear Solver: A Hybrid Algorithm for Linear Systems

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## Abstract

Previously proposed quantum algorithms for solving linear systems of equations Ax = b cannot be implemented in the near term due to the required circuit depth. Here, we propose a hybrid quantum-classical algorithm, called Variational Quantum Linear Solver (VQLS), for solving linear systems on near-term quantum computers. VQLS seeks to variationally prepares x such that Ax is proportional to b.

## Cost Functions and Circuits

We present efficient quantum circuits to estimate C, while providing evidence for the classical hardness of its estimation.

$$|\psi\rangle := A|x(\theta)\rangle$$

**Global cost**

$$\hat{C}_G := \mathrm{Tr}[|\psi\rangle\langle\psi|(I - |b\rangle\langle b|)]$$

$$C_G := \hat{C}_G / \langle\psi|\psi\rangle = 1 - |\langle b|\psi\rangle|^2$$

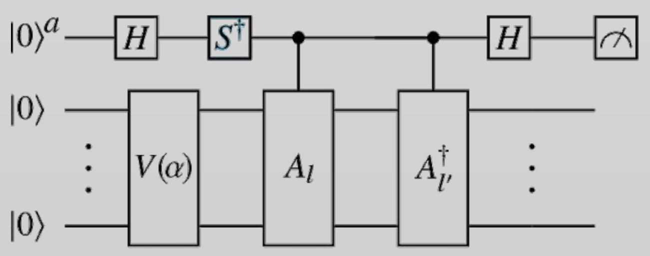
**Local cost**

$$H_L = A^\dagger U \left( I - \frac{1}{n}\sum_{j=1}^{n} |0_j\rangle\langle 0_j| \otimes I_{\bar{j}} \right) U^\dagger A$$

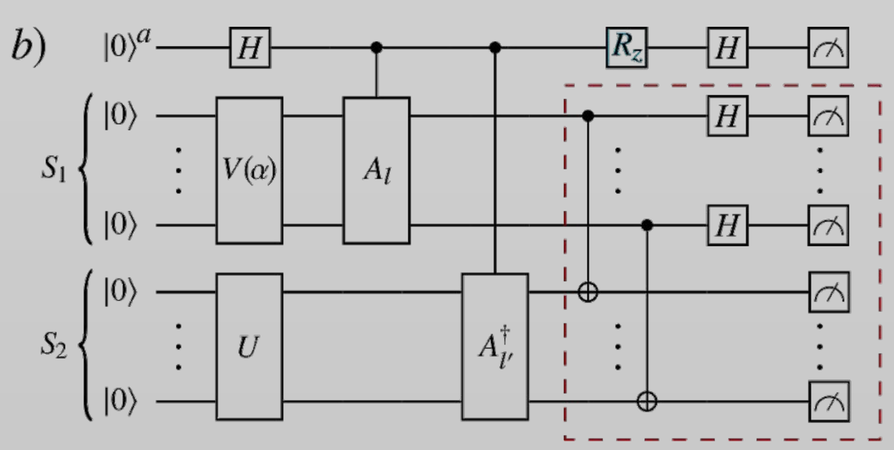
$$\hat{C}_L := \langle x(\theta)|H_L|x(\theta)\rangle$$

We prove that the problem of estimating any cost function to within polynomial precision is DQC1-hard, and that

$$C_L \leq C_G \leq n C_L$$



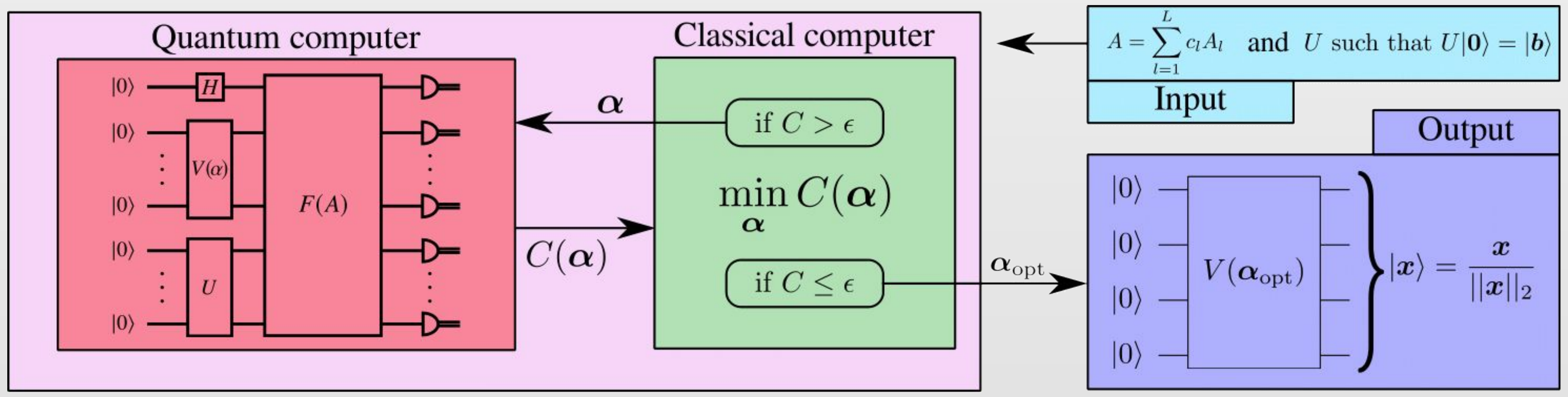
Hadamard Test circuit to compute one of O(L^2) terms in the global cost.



Novel Hadamard Overlap Test circuit to compute one of O(L^2) terms in the global cost function.

## Algorithm Overview

VQLS assumes that A is given as a linear combination of unitaries; we provide an efficient procedure for obtaining this decomposition when A is sparse. Furthermore, we derive an operationally meaningful termination condition for VQLS that allows one to verify that a desired solution precision, epsilon, is achieved. Specifically we prove that C >= epsilon^2 / kappa^2, where C is the VQLS cost function and kappa is the condition number of A.

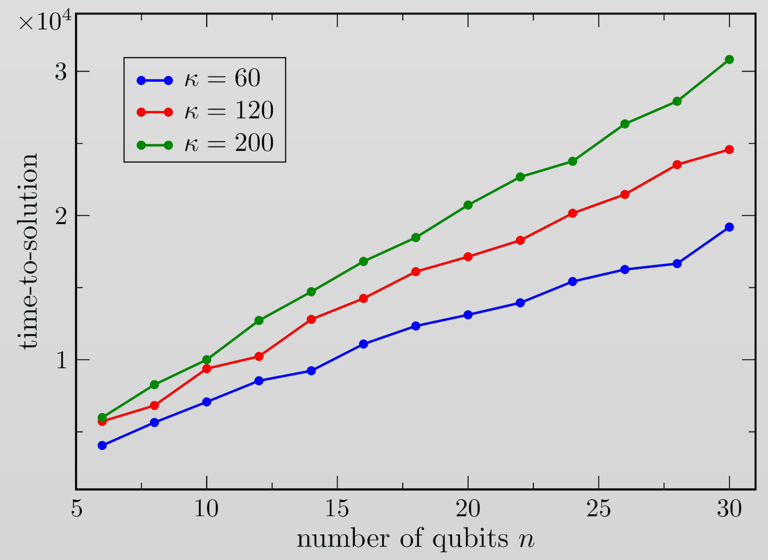



## Scaling Heuristics

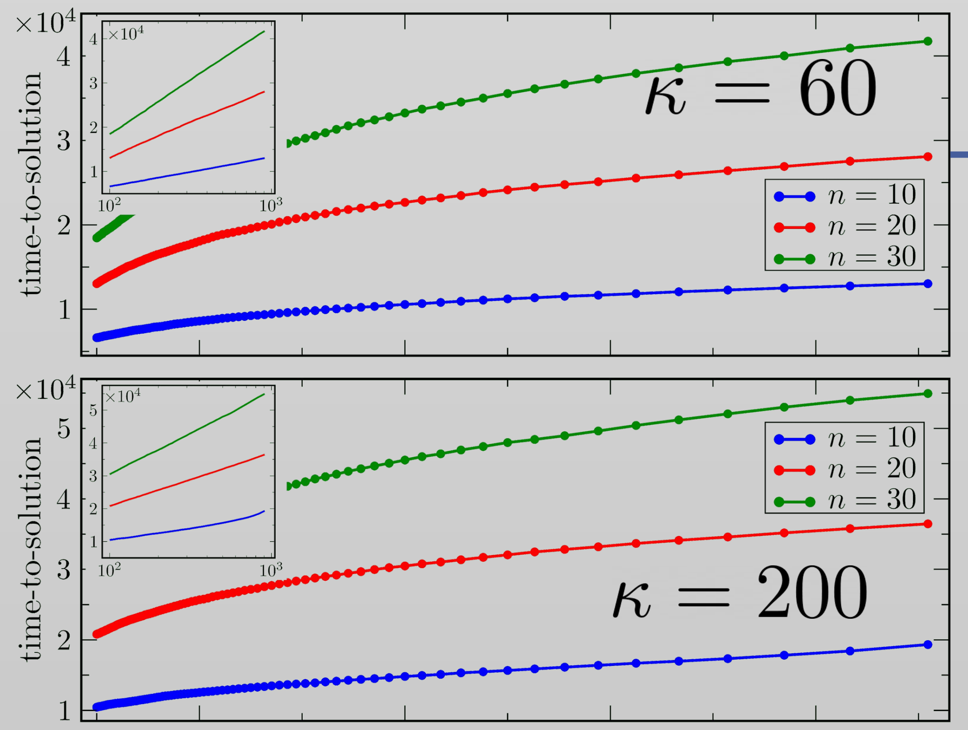
For the specific examples that we consider, we heuristically find that the time complexity of VQLS scales efficiently in the desired precision epsilon, condition number kappa, and size of the system N.

$$A := \frac{1}{\zeta}\left(\eta I + \sum_{j=1}^{n} X_j + J\sum_{j=1}^{n-1} Z_j Z_{j+1}\right) \qquad |b\rangle = H^{\otimes n}|0^{\otimes n}\rangle$$

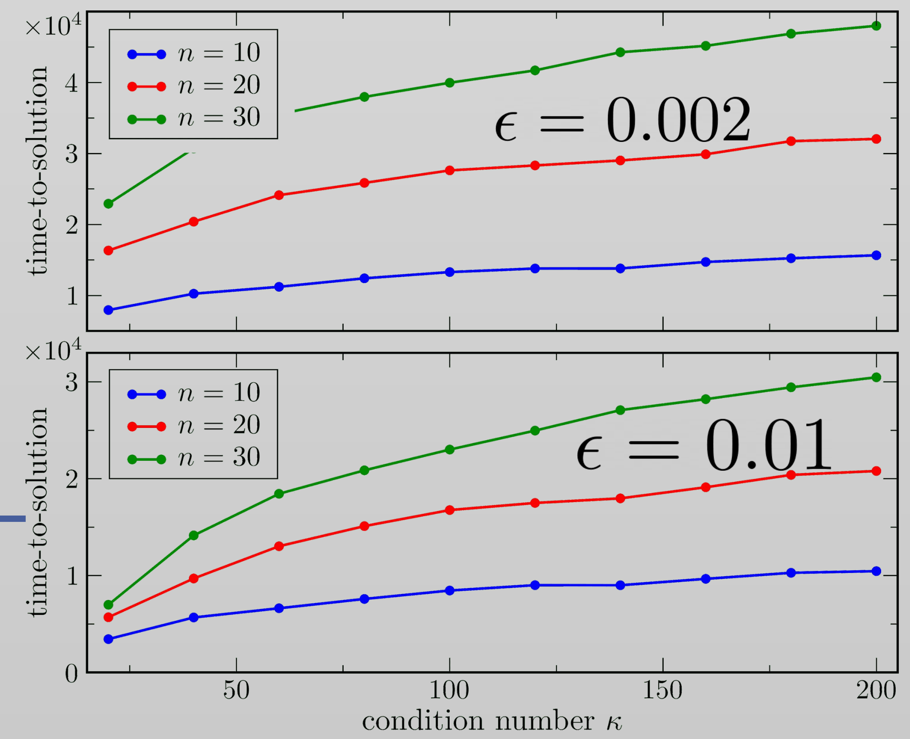
**Scaling in size of Ising model linear system**




**Scaling in precision of solution**




**Scaling in condition number of matrix**

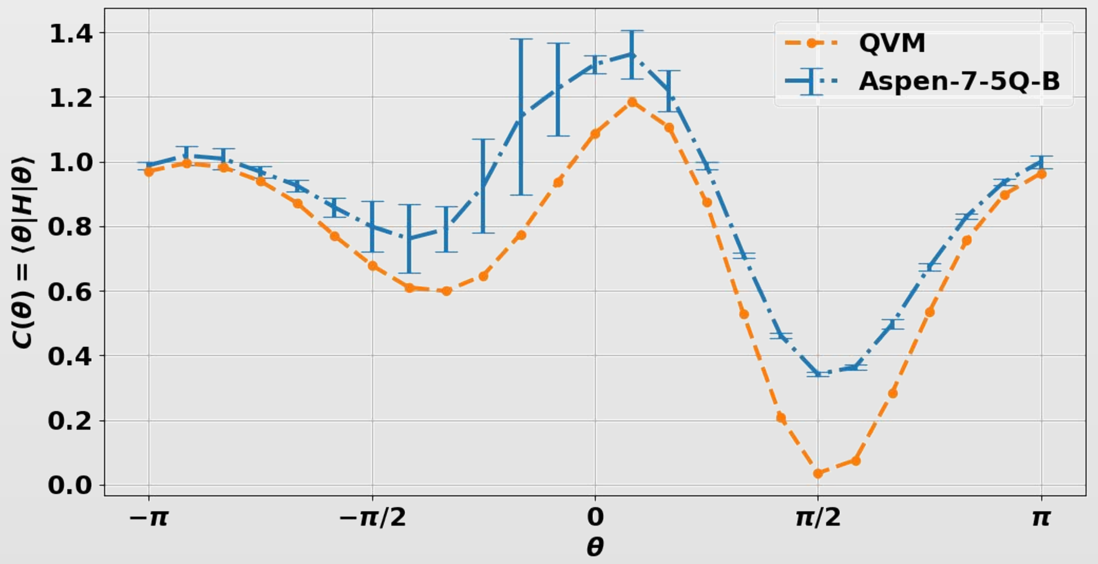



## QPU Implementations

For implementations on quantum processors, we minimize the energy of the effective Hamiltonian

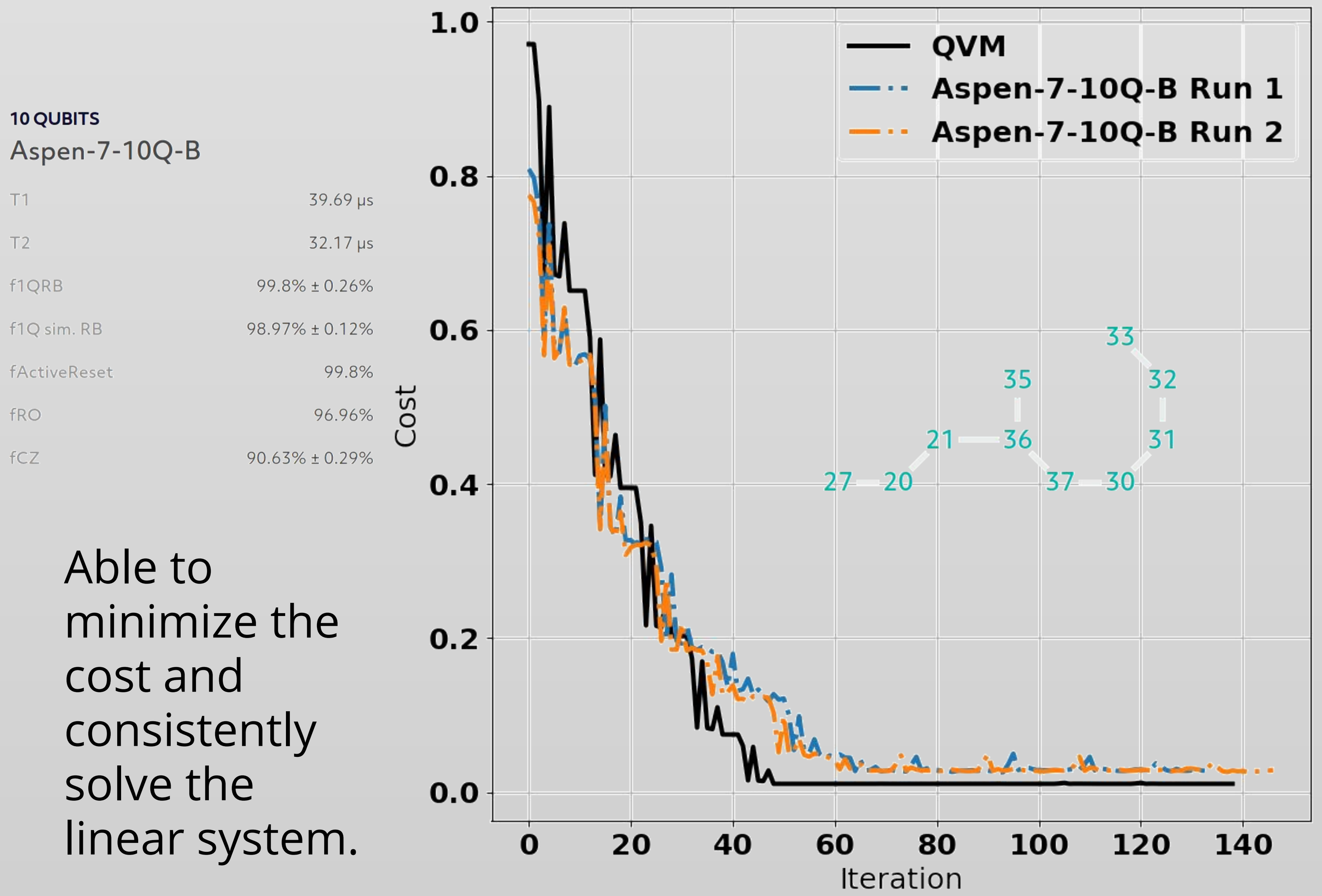
$$H_{A,\mathbf{b}} := A^\dagger(I - |\mathbf{b}\rangle\langle\mathbf{b}|)A$$

$$A = I + 0.2X_1Z_2 + 0.2X_1 \qquad |b\rangle = H^{\otimes n}|0^{\otimes n}\rangle$$




Cost landscape on QPU matches cost landscape on simulator well.

Using Rigetti's Aspen-4 and Aspen-7 quantum computer, we successfully implement VQLS up to a problem instance of size 1024 x 1024.




Able to minimize the cost and consistently solve the linear system.

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